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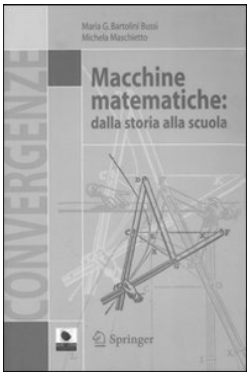
Didactics

# *Mathematical Machines: A Laboratory for Mathematics*

**Abstract.** *Macchine matematiche; dalla storia alla scuola* is a book by Maria Grazia Bartolini Bussi and Michela Maschietto that provides a description of the reconstruction of many historical machines for drawing, and their impact on the history of mathematics. Underlying the book's thesis are ideas about education and developments in neuroscience. Laura Tedeschini Lalli discussed the book and the educational experiments and their goals with author Maria Grazia Bartolini Bussi.

## *Introduction*




The Festival of Mathematics, which has taken place at Rome's Auditorium Parco della Musica for the past three years (this year's edition took place 19-22 March 2009), draws many visitors of all ages from all over. This year I met many architects there who came partly out of general curiosity, but also to see the exhibit entitled "Mathematical Machines" curated by the research group of the same name from the University of Modena and Reggio Emilia, coordinated by Mariolina Bartolini Bussi. The Roman architects were fascinated by the possibility of seeing up close how the mathematical machines work; the exhibit featured instruments designed centuries ago, reconstructed down to the smallest detail, and workable.

The book *Macchine matematiche: dalla storia alla scuola* (Milan, Springer Italia, 2006, 159 pp.), one of the books in the series entitled "Convergenze" published by the Unione Matematica Italiana through Springer Italia, provides a description of many of the reconstructed machines used for drawing, and their impact on the history of mathematics. The first half is dedicated to the instruments and the mathematical laws behind them, and gives the formulations of some of the "geometric problems" that they solve graphically. All of this is presented together with historical notes. The second half discusses educational experiments performed in schools on different continents, intended to help build the capacity for abstraction through knowledgeable use of the drawing machines.

The lengthy bibliography is divided according to topic, and distinguishes historical sources from modern works about history. It also includes educational experiments and works on teaching mathematics and the capacities for abstraction.

The book comes with a CD that contains animations of the drawings. It also contains other interesting documents, such as the policy of the Unione Matematica Italiana regarding a laboratory of mathematics in schools, whether such a laboratory is possible and its nature.

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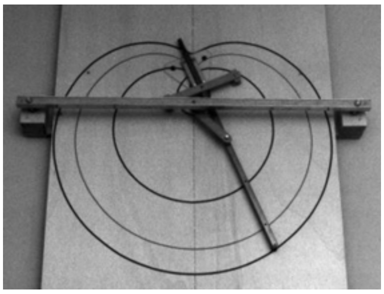


Fig. 1. Generation of fourth-degree algebraic curves, Pascals' snails

The historical approach is very nice, for non-mathematicians as well as mathematicians, with an entire chapter dedicated to a detailed comparison of Desargues and Descartes and their quite different models of rationality. The historical approach also makes it possible to look at modernity and the future, telling how branches of mathematics developed and were generalised from the line of reasoning that grew up around the mathematical machines for drawing, branches that are still fertile and today are abstract, elegant and apparently disconnected from the need for graphic representation.

I interviewed Mariolina Bartolini Bussi to learn more about her experiences with and thoughts about the machines and, more generally, about teaching mathematics.

**Laura Tedeschini Lalli**: The methods and aims of this research are typical of the Italian outlook; they are historical studies intended to regain possession of methods of teaching and learning.

**Maria Grazia Bartolini Bussi**: The project about machines was born of our interest in history, but from the very beginning it had an educational purpose, because the machines were to be used by students. This is different from what is usually done with historical or museum projectcs. For example, in the book we mention the 2001 exhibit "Nel segno di Masaccio" at the Uffizi in Florence, which featured original drawings and valuable instruments designed by engineers and built by specialised craftsmen. But these recreations were so precious that no one could touch them, and as a final result of a research project this was inconceivable for us. The machines had to be usable in order for them to be understood.

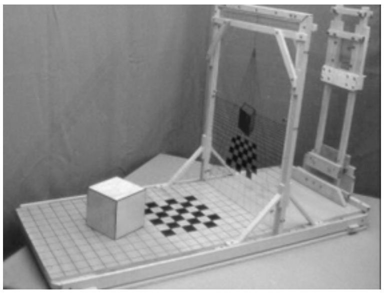
**TL**: Wasn't there anyone at the exhibit who was authorised to use the machines?

**BB**: No, not even the guides, who were only there to make sure that the visitors didn't touch them. I myself had used gestures to explain to my companions how the machines worked without touching them; the guides came up to listen, astounded at how much I knew about the machines, so I explained that we had a lot more machines like these, and that ours could be used, and so I knew them quite well. As I said, in the beginning we were interested in the historical value of the machines, but also in their educational value because one of our questions is, how can we communicate to students who live in a world that is so virtual, instantaneous and disconnected from history that they too are part of the unfolding of history? It might be possible to do this if they can use objects that themselves have a history. There is an educational message in our use of history: be proud of your culture because, for example, Italy has produced this kind of thing. Today we hear a lot about intercultural problems, but in order to speak about interculturalism we need to know what a culture is and love our own. Then later, loving our own culture, we can also open up to the world. I say this as someone who trains teachers, and I speak from having worked with and thought about these problems for years. So it's important to me to underline the epistemological aspect in the fullest sense of the term, that is, the epistemology of the building up of knowledge, even today. There is a cognitive aspect here that is quite strong, and harmonises with the results of recent research on the powerful physical aspect of experience, the fact that knowledge, even abstract knowledge, is embodied. This has also been confirmed by neuroscientists.

TL: This concept is quite explicitly expressed in your book as well. Can you tell us more about the results of the research of neuroscientists regarding abstraction?

BB: We actually maintain a certain distance from this aspect: research undertaken by very capable cognitive linguists in San Diego has unfortunately neglected the instrumental aspect. To me the subject is not a naked man facing the world; the subject is a man who lives in a culture that has produced artifacts, and the knowledge that has been built up is mediated by these artifacts. This does not appear in the analyses performed by cognitive linguists. The only artifact that they have taken into consideration is the one that, of course, they cannot exclude, which is language. Instead, we want the material aspect to be considered as well. If we take the next step and move to neuroscience, then I'll only note that the time required and complexity of the processes for learning mathematics are still too far removed from those studied using methods of neuroimaging. As far as materiality is concerned, you and I share this view, don't we?

TL: Yes, I teach mathematics in a school of architecture, and I know that the architecture students are skilful at thinking with pencils in hand. One of the studies that interests me greatly is the role of the physical-mathematical model in reinforcing a cultural formulation. This is difficult to discuss, because so often it leads to misunderstandings. Sometimes the physical-mathematical model is implied, and appears to be a in tool for measuring, because it underlies and governs the design of the tool. I would imagine that it is similar for instruments for drawing.



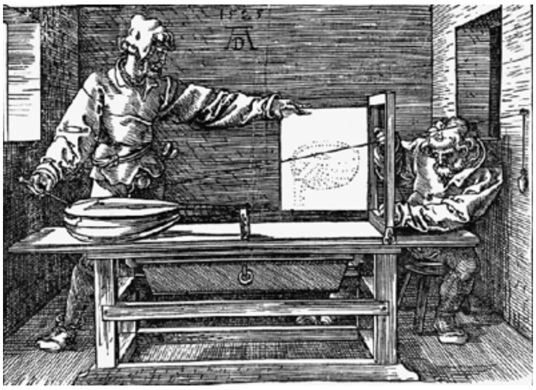


Fig. 2. above) Dürer's perspectograph, the "veil"; right) Albrecht Dürer drawing on glass (c.1520), etching

BB: When we talk about models we always have to add an adjective, that is, sometimes we mean mathematical models, sometimes physical models. Since we are involved with mathematical machines, we naturally speak of "models" in both senses: we have the mathematical model, and we also have a physical model of the mathematical model, that is, of the law that underlies it. We believe in this cultural approach, because it comes from history, and adds value to the aspects of mathematics that are tied to other aspects of human knowledge in history. We also make models, but this is a complementary approach. Sometimes the mathematical machines intentionally demonstrate mathematical laws that are already known (such as in the case of many of the machines for drawing curves).

Sometimes they precede the formulation of mathematical laws (as in the case of perspectographs).

**TL**: It seems to me that it is important to bring to conscious awareness the existence of an implied model. I see that you too share this conviction, because in the book you discuss experiments that you call "black box experiments", which I also make with my students. I call them "inverse problems", although perhaps this is too strong a term, because they are actually an initiation into the class of inverse problems. You give two drawn versions of the same figure, and ask which mathematical machine, or which procedure, changed one into the other. Asking which machine is used to create the representation already sets the stage for recognition of the model.

**BB**: It's true. We are a group of researchers working in the teaching of mathematics, and thus we study what effects the intentional use of these mathematical machines have on processes of teaching-learning. We began in schools at the pre-university level, although more recently we have also begun to transpose the results to the university level, for teacher training for elementary schools and secondary schools.

**TL**: Many architects have studied theories and rules for graphic representation that were born around these machines without, however, actually using them. In your book you mention the collection at Cornell University. Do they allow their machines to be used like you do?

**BB**: Cornell has the collection of Reuleaux (1829-1905), who had reconstructed more than 800 fundamental mechanisms, the simple machines. They have a very interesting virtual library (http://kmoddl.library.cornell.edu/index.php). They also have there an experimental project: Daina Taimina, a Latvian who lives there, presents the machines that most lend themselves to manipulation at schools in the area. This kind of approach is related to the idea of a laboratory of mathematics as a laboratory for experimentation. Mathematics too has an experimental component which favours it most creative aspects, that is, the genesis of conjecture, and construction. In our schools, above all when dealing with something in three dimensions, everything is flattened in order to replicate the proofs that the teacher has shown on the blackboard or are shown in the textbooks. Never, or hardly ever, do we see the moment of discovery or invention. In one of the courses for teachers that I teach at the university, I spend a lot of time on this creative aspect. Take the machines, for example: some have furnished the basis for the lines of reasoning of geometers of the 1600s who did not have continuum theory at their disposal; so what was a continuum for them at that time? It was what could be described by motion. In spite of this they invented valid theorems and formulated proofs which, according to the standards of rigour of the times, were rigorous. Today we have dynamic geometry software which they didn't have and which, at least in the plane, makes things easier. We recuperate the aspects of thought that are creative, inventive, and which otherwise schools would tend to neglect: the genesis of predictive hypotheses (how the machine will work); of interpretative hypotheses (why does a given machine work that way?). We do this work both when the machines are present and when they are not, that is, with the help of mental experiments. We think that these things can and must be done from the time that the students are very young, and we have had very positive responses even at the elementary school level. But if the teachers don't themselves have experience with these processes, then only rarely are they able to propose this kind of experiment. In our elementary schools the students arrive at *reductio ad absurdum* proofs in a completely natural way by making objects move (mentally, without touching them) and they can see that these objects cannot move, because that would be contrary to the rules that they themselves discovered and memorised earlier. It is not the impossibility of motion that they discover, but the theoretical impossibility of motion. This is the most interesting aspect of our experiments: we have proven that this can be done with children eight, nine, ten years old, in the usual time required for learning. Everything is done through touch, and above all through the intelligent use of language, other semiotic systems, signs, drawings, gestures, words, reasoning. In the book we deal less with the educational results: it is a mathematics book, and thus it deals not only with the aspect of education but with those of history and mathematics as well. We were asked to write a "small" book, not an encyclopedic treatise, one that would be of interest to a diverse group of readers, but above all to teachers.

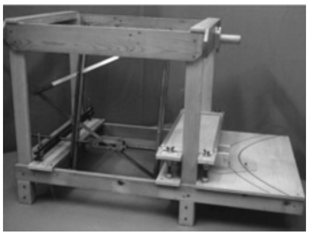


Fig. 3. Descartes's machine for hyperbolic lenses

**TL**: On the other hand, representation by means of manipulation and abstraction is interesting to a wide range of people. How has it been working with your experiments in places that are far apart both culturally and geographically? I like to say that "our eyes are culturally trained"; all of us, even those from other countries who come to Italy to study, have eyes that are culturally trained by the same visual culture. Part of this visual culture, for example, is one-point perspective. When eyes are not culturally trained with regards to the same artifact, what happens?

**BB**: We hardly ever perform the experiments directly ourselves. They are usually performed by our colleagues who live there with whom we have collaborated. In some case doctoral theses are involved; sometimes our experimental protocol has been modified. These activities have generally worked very well everywhere. We have different forms of collaboration on all five continents. Sometimes our colleagues have begun their own work, then learned about ours and contacted us, leading to us working together, especially on articulated systems (machines to lay out curves and pantographs), that is, on the more technological aspects. We haven't always pursued the same objectives, and we have tried, more briefly in the book and more in-depth in the CD, to report on the experiments. Naturally, the aspect of perspective is much more typically Italian, closer to our own culture.

**TL**: Even closer to Italian culture than to European culture?!

**BB**: As far as perspective is concerned, not even in Europe have we found researchers to collaborate on experiments with us: the few experiments on perspective that have been done, we have done ourselves. It might be that such experiments are made in schools of architecture, but we are not aware of any. Others have worked on articulated systems, on machines to lay out curves, pantographs, machines for transformations: these are somehow universal. I personally followed some phases of experiment in Mexico, but this was a school based on a European system and the instructors had in any case earned their Ph.Ds in Europe. The instructors in Mali and Mexico that we talk about in the book earned their degrees in Grenoble. So in some way in order to be part of a project like ours the researcher has a kind of cultural conditioning; the students may not, but the researcher does. In other words, we have had a good response from schools that can be defined as Western, even though they might be in Africa: they are schools in any case that follow a European model. Many wonderful experiments have been performed in Australia, including experiements with students who are quite young.

TL: Have you come across cultural differences in all your long experience?

BB: Without a doubt, cultural differences exist and become evident. Rarely do they emerge when using an artifact, a machine: an artifact already embodies knowledge. More often they come out when natural phenomena are concerned, for example, shadows. We have worked a great deal with shadows from the sun; some of our machines allude to shadows, they are models (physical models of mathematical models) of shadows, but they deal with the mathematical model, built with our Western mentality. We have had collaborators who have performed experiments in African, in Italian schools in Ethiopia and Eritrea. Even though they go to an Italian school, the children are not Italian, and they have perceptions and a system of beliefs about shadows that are very different from those of Italian children. The shadow is a very thorny object, above all when the shadow of one's own body is involved, because it is our "double". As long as we study shadows of sticks and neutral objects, all is well, or at least, it is easier to construct the mathematical object. Of course, the cultural aspect has to be taken into account. We are being pressed to translate this book, most likely into English. I don't think, however, that this book is universal. I am a member of the International Commission for mathematics teaching and I assure you that we come up against cultural differences as well as differences in models of rationality. Mathematics is not "simply" universal, as many like to think. Or better, mathematics as a crystallised product can be universal, but the process of approaching mathematics might not be the same in all cultures.

TL: Can you tell us more about cultural differences and convergences and mathematics?

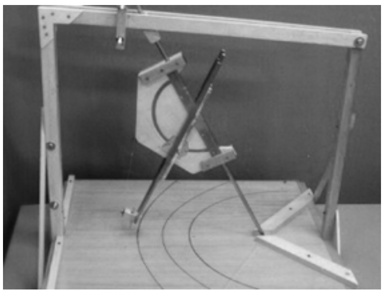


Fig. 4. A perfect compass

BB: One of my Japanese colleagues worked on antique Japanese books and found articulated systems very similar to ours. They are the same as our instruments for laying out continuous curves from the 1600s. One article written by six scholars, including a Japanese and a Latvian, also retraced the European influence both in the East, in Japan, and in the West, in the United States, using the emblematic case that we cited of the Cornell collection.

However, in ancient Chinese technology there were machines that were extremely similar to ours, which they used for things that are extraordinary to us, such as realising etchings on jade.There are aspects of technology that are shared by many countries.Instead, we have a situation which is rather exceptional where perspective is concerned. The birth of perspective as a science is rather recent, as far as we know, and so we can follow it from its inception. Straightedge and compass were born somewhere at the dawn of time, that is, we know very little about what geometry was before Euclid, but on the other hand it is possible to reconstruct the birth of perspective. In this case, technology – the machines – came before the mathematics, because the theory of transformations and projective geometry developed later. Many mathematicians think of mathematics as the queen of sciences: that pure mathematics comes first and applied mathematics only later. Instead, quite often fundamental theories of mathematics, such as projective geometry, were born from applications. We have many examples to counter this widespread belief. So, we can use the machines to break the stereotype: mathematics is only a product of man's culture, which has dialectic relationships with the other products, before, after and sometimes at the same time; sometimes it is separate. We get the impression of a history of mathematics that is sometimes too cut and dried. History can help us quite a lot to see the world and the evolution of the building up of knowledge. If we read some of the authors of the 1600s, such as Desargues, we come to see that the way of thinking is not that of Descartes, who is considered today by many to be *the* model for rational thought. The complex aspects of our history can help us to accept the diverse models of rationality that have contributed to the construction of mathematical meanings, even at the cost of renouncing rigour. René Thom, winner of the Fields Medal, once said that if he had to choose between meaning and rigour, he would unhesitatingly choose meaning. If we were to interview the great mathematicians of today about their creative processes, they would confess that in reality the creative process isn't at all a deductive, rigorous process: it goes forward by intuition, which is then adjusted as it goes along. Only the mathematicians with more modest gifts work deductively.

TL: Can we say that, in formulating hypotheses, you emphasise the inductive moment?

BB: In addition to the inductive process, in many cases there is another process, described by Peirce, that of abduction: recognising something that you might have already seen and being able to connect it to other systems of knowledge, being able to make a conjecture which has still to be proven, but which already has a higher degree of probability of being valid. I sometimes say to my university students, "Look at this figure: do you recognise something?" They might close their eyes, in order to see it better in their minds, then suddenly they open them again and begin to laugh: they recognised something! This is an abduction, recognising in this complex figure something that they already knew. It is clear that the more things you know, the easier it is to make abductions. If you know only straight lines and conics, then you see an arch and either think of a straight line or a conic, or nothing at all comes to mind. This year we looked at some third- and fourth-degree curves, studying their mechanical generation, and they are now able to recognise those too, to produce hypotheses. So they have increased their creative potential, because knowing more things, they are able to recognise, in complicated situations, more things that are related to the objects that they have constructed. Creativity is also due to the quantity of knowledge that you have internalised: the more you know, the more creative you are. The more things you know, the more things you can be creative about, carrying them from one area to another. So, in addition to deduction and induction, there is also abduction.

TL: What were your experiences with a public that is adult and different from that of the schools, and with architects in particular? They find this idea of reconstructing machines to be particularly interesting.

**BB**: My impression, and this was confirmed at the Festival of Mathematics in Rome, is that everyone sees something different. One can see concretely the objects that have been studied; there is probably an aesthetic aspect as well, because our machines are beautiful. There is the potential to interest a quite diverse public. In Modena we had a seminar with those interested in philosophy and letters. However, I see that the architects, who up until now we have not worked with a great deal, might be or should be among our favourite interlocutors. Alessandra Mariotti and I have been in contact with Gabriela Goldschmidt, an Israeli architect who was taking a sabbatical year at MIT. On that occasion we were invited to MIT, because she had so appreciated these machines. They had briefly involved us in an international group working on cognitive processes in design, which was very interesting. They studied the processes used by architects to design and create new projects; it was tied to the study of creativity. We saw how they worked, I would call it cognitive architecture: they collected notebooks, conducted interviews, showed architects who reasoned out loud, in order to understand how a design is put together, what ways and means are used. We found it interesting that there were aspects that we also find in the mathematical way of thinking: a continual oscillation between overall intentionality and a return to very small, almost microscopic aspects of design. But the experience lasted only a short time. I remember that the dean of the school of architecture at MIT didn't miss a word, in the end.

*Translated from the Italian by Kim Williams*


### *Acknowledgment*

The images all come from the website of the Laboratorio Macchine Matematiche of the Università di Modena e Reggio Emilia.
http://www.museo.unimo.it/theatrum/macchine/_00lab.htm



### *About the author*

Laura Tedeschini Lalli, Ph.D., is full professor of Mathematical Physics in the Facoltà di Architettura of Università Roma Tre. Her formal training is in mathematics and in musical composition. She received her Ph.D. in applied math from the University of Maryland in College Park. Her research is in chaotic dynamical systems (theoretical) and some applications; in sonification of data; in the (often implicit) role of the mathematical model in the communication of science. She is among the founders of European Women in Mathematics (EWM). She coordinates the Laboratorio di Matematica di Roma Tre, where architects and mathematicians work together.